J. Appl. Maths Mechs Vol. 56, No. 3, pp. 346–358, 1992
Printed in Great Britain.

0021–8928/92 $15.00 + .00

ON THE PROPAGATION OF DISCONTINUITIES IN A DRIFTING ICE COVER†

A. V. MARCHENKO

Moscow

(*Received* 14 *February* 1991)

A drifting ice cover on the surface of the sea is considered. The ice cover consists of ice floes having various sizes, shapes, and strength properties. For sufficiently rapid and small compressive-tensile loads, each ice floe behaves as an elastic body and its deformation can be described by the model of a linearly elastic Hooke body. At higher loads, the ice floes break-up [1].

If the ice floes are located uniformly on the surface of the water and the relative velocities of adjacent ice floes are small, the motion and deformation of the ice cover can be described as a continuum with a visco-elastoplastic rheology [2–4]. The plastic properties are associated with irreversible changes in the ice cover due to the shifting and breaking-up of separate ice floes as they interact and form into hummocks. The viscous properties are manifested when inelastic collisions between ice floes become the main form of interaction in a particular area of the sea surface; such collisions occur where the drift velocity gradients are high and the ice cover is sufficiently sparse. Elastic stresses may arise in a compacted ice cover.

A model of an ice cover with elastoplastic rheology is proposed. One-dimensional discontinuous solutions of the model equations are considered. The problem of the collision of two ice fields of different compactness and the problem of condensation of a drifting ice cover near a solid wall are solved.

1. CONSIDER an ice floe floating on the surface of water. There are no surface waves in the water and the ice floes move in the horizontal plane. The main forces that impell the ice floes are forces of atmospheric and oceanic origin and the forces of interaction with surrounding bodies.

We define the Cartesian coordinate system x_1, x_2, z, where z is directed vertically upward. The equations of continuity and momentum, integrated across the thickness of an ice floe, are written in the form

$$\frac{d}{dt}\int_{S_f}\rho h\,ds+\int_{\partial S_f}\rho h(\mathbf{u}\cdot\mathbf{n})\,dl=0 \tag{1.1}$$

$$\frac{d'}{dt}\int_{S_f}\rho h\mathbf{u}\,ds+\int_{\partial S_f}\rho h\mathbf{u}(\mathbf{u}\cdot\mathbf{n})\,dl=\int_{\partial S_f}\mathbf{F}_n\,dl+\int_{S_f}\mathbf{F}\,ds$$

$$d'/dt=d/dt+\boldsymbol{\Omega}\times,\quad \boldsymbol{\Omega}=(0,\ 0,\ \Omega)$$

Here ρ is the thickness-averaged density of the ice floe and h is the thickness, $\mathbf{u}=(u_1, u_2, 0)$ is the velocity vector of ice particles averaged over the ice floe thickness, Ω is the Coriolis parameter, S_f, ∂S_f are an arbitrary area on the ice floe and its boundary in the plane $z=0$, $\mathbf{F}_n$ is the force applied to the part of the boundary with the outer normal $\mathbf{n}$ and $\mathbf{F}$ are the external forces exerted by the atmosphere and the ocean.

System (1.1) is not closed and it must be augmented with rheological relationships that define the dependence of $\mathbf{F}_n$ on the strain parameters of the ice with boundary conditions on the edge of the ice floe. Note that the nature of the forces acting on the edge of the ice flow may be different from the stresses within each floe.

† *Prikl. Mat. Mekh.* Vol. 56, No. 3, pp. 419–433, 1992.

2. Consider an ice field floating on the surface of the sea and consisting of separate floes that may interact with one another. The motion and the deformation of each floe are described by Eq. (1.1).

In the plane $z = 0$ define the function $f(x_1, x_2, t)$ by the following rule:

$$f = \begin{cases} 1, \text{ if there is ice at point } (x_1, x_2), \\ 0, \text{ if there is no ice at point } (x_1, x_2). \end{cases}$$

Multiplying both sides of Eqs (1.1) by f, we obtain

$$\frac{d}{dt}\int\limits_S \rho h f\,ds + \int\limits_{\partial S} \rho h\,(\mathbf{u}\cdot\mathbf{n})\,f\,dl = 0 \tag{2.1}$$

$$\frac{d'}{dt}\int\limits_S \rho h\mathbf{u} f\,ds + \int\limits_{\partial S} \rho h\mathbf{u}\,(\mathbf{u}\cdot\mathbf{n})\,f\,dl = \int \mathbf{F}_n f\,dl + \int\limits_S \mathbf{F} f\,ds$$

Here S is an area in the plane $z = 0$ and ∂S is the boundary of this area. If $f \not\equiv 1$ on S, then there are both ice floes and clear water space in S.

Equations (2.1) differ from (1.1) in that the expression for $\mathbf{F}_n$ in (2.1) contains forces of interaction between ice floes.

Partition the area S into elements s_α and the boundary ∂S into corresponding elements l_β. Applying the mean-value theorem to (2.1), we obtain

$$\frac{d}{dt}\sum_\alpha (\rho h f)_\alpha s_\alpha + \sum_\beta (\rho h f\,(\mathbf{u}\cdot\mathbf{n}))_\beta l_\beta = 0 \tag{2.2}$$

$$\frac{d'}{dt}\sum_\alpha (\rho h f\mathbf{u})_\alpha s_\alpha + \sum_\beta (\rho h\mathbf{u} f\,(\mathbf{u}\cdot\mathbf{n}))_\beta l_\beta = \sum_\beta (\mathbf{F}_n f)_\beta l_\beta + \sum_\alpha (\mathbf{F}\cdot f)_\alpha s_\alpha$$

Equations (2.2) contain the values of the discontinuous functions ρhf, $\rho hf\mathbf{u}$, ... averaged over s_α and l_β. If the characteristic scales of variation of the mean values of ρ, h, f, $\mathbf{u}$, $\mathbf{F}_n$, $\mathbf{F}$ substantially exceed the horizontal scale of s_α, then s_α may be treated as the element dS of some continuum on which sufficiently smooth functions ρ, h, A, $\mathbf{u}$, $\mathbf{F}_n$, $\mathbf{F}$ are defined; the values of these functions on dS are equal to their mean value on the corresponding element s_α. The function $f(x_1, x_2, t)$, is the mean value of $A(x_1, x_2, t)$ and it is called the compactness of the ice cover. Note that the ice floes included in the element dS may be of different thickness. The mean thickness is assumed to change slowly on passing from the element dS to adjacent elements.

Contrary to classical statistical physics [5], this model does not require that the element dS contain a large number of ice floes. This condition is replaced with the condition of uniform distribution of ice floes over the areas s_α and small scatter of their relative velocities. This is attributable to the strong dissipation of energy in the ice cover–water system due to the interaction of the ice floes with water and with one another through inelastic collisions [1, 2]. If the forces $\mathbf{F}$ impelling the ice do not change within the area element dS, then eventually the velocities of all ice floes in dS are equalized.

Under our assumptions, Eqs (2.2) may be rewritten in the form

$$\frac{d}{dt}(\rho hA) + \rho hA\nabla\cdot\mathbf{u} = 0, \quad \rho hA\frac{d'}{dt}\mathbf{u} = A\mathbf{F} + \nabla\cdot\sigma \tag{2.3}$$

$$\frac{d'}{dt} = \frac{\partial}{\partial t} + \mathbf{u}\cdot\nabla + \Omega\times, \quad \nabla = \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, 0\right)$$

$$\sigma = \|\sigma_{ij}\|, \quad \sigma_{ij}n_j = AF_{n,j}, \quad (i, j) = (1, 2)$$

To construct the continuum model of the ice cover, we need to define the relation between the stresses σ_{ij} and the strain parameters that reflect the physics of the interaction between ice floes and the deformation of each ice floe separately. In constructing the rheological relationships of the model, we will consider only fairly rapid processes, when stress relaxation can be ignored and each ice floe may be regarded as a linearly elastic Hooke body under small loads [1, 2]. As the loads increase, the ice floe breaks-up.

From the equation of continuity and the definition of the strain tensor $\varepsilon_{ij} = (\partial w_i \partial x_j + \partial w_j/\partial x_i)/2$ we obtain

$$2\varepsilon = 1 - \rho_0 h_0 A_0/(\rho h A), \quad \varepsilon = -{}^1/_2 \varepsilon_{ii} \tag{2.4}$$

The variables with a subscript of zero correspond to the parameters of the ice cover in the undeformed state.

It follows from (2.4) that tensile-compressive strains of the ice cover may produce changes in ρ, h and A. We will divide these strains into reversible and irreversible (plastic) strains. Reversible elastic strains change ρ, leaving h and A constant. Plastic strains are subdivided into condensation (compaction) and hummocking. Condensation produces changes in ρ and A, while h remains fairly constant. Hummocking changes all ice cover parameters ρ, h and A. For elastic strains and condensation, we respectively obtain from (2.4)

$$2\varepsilon = 1 - \rho_0/\rho, \quad 2\varepsilon = 1 - \rho_0 A_0/(\rho A) \tag{2.5}$$

3. We identify four phase states of the ice cover: (a) dispersed, (b) compacted and (α) formed into hummocks and (β) not formed into hummocks. Ice in states (a) and (b) may be formed into hummocks or not. Collisions are the main form of interaction between ice floes forming a dispersed ice cover. Each ice floe floats on the surface of the water so that it can move only inside a certain neighbourhood without touching the nearby ice floes. The interaction of ice floes forming a compacted ice cover involves mutual compression and friction at points of contact. Note that in reality the state of the ice cover may be a mixture of the above-mentioned basic states, in which case we accordingly consider the predominant type of interaction of ice floes in the observed section of the ice cover.

The following phase transitions are allowed: (a) (α), (β)$\rightleftarrows$(b)(α),(β),(b),(β)$\rightarrow$(b)(α). The transition (α)$\rightarrow$(β) is not allowed. The transition (a)$\rightarrow$(b) is obviously associated with compressive strains. It can be compared with the formation of a solid porous skeleton in soils [6]. We assume that for any section of a dispersed ice cover the transition to the compacted state occurs at a certain compactness value A_* that depends on the shape, the size, and the relative location of the ice floes in that section. The transition (b)$\rightarrow$(a) occurs under tensile strains for $p = -\pi_b \leqslant 0$ ($p = -{}^1\!/\!_2\sigma_{ii}$). The pressure $\pi_b \neq 0$ when the ice floes are linked.

We will write the hummocking condition for a compacted ice cover in the form

$$|\tau_n| = f_t(\sigma_n, A, h) \tag{3.1}$$

where τ_n and σ_n are the tangential and the normal components of the stress on an area element with the normal $\mathbf{n} = (n_1, n_2)$.

For a dispersed ice cover, we take

$$\sigma_{ij} = dh/dt = d\rho/dt = 0 \tag{3.2}$$

System (2.3) and (3.2) is closed and it describes the behaviour of the ice cover for $A < A(\xi_1, \xi_2)$, where $\xi_{1,2}$ are the Lagrangian coordinates of the ice cover element.

Loading of a compacted ice cover produces both elastic and plastic strains. The plasticity condition is taken in the form

$$|\tau_n| = (\sigma_n - \pi_b)\operatorname{tg}\gamma(A, h) \tag{3.3}$$

$$p = \pi_p(A, h) \tag{3.4}$$

where γ and π_p are the angle of internal friction and the pressure for which $dA/dt > 0$, $dh/dt = 0$ for $A < A_{**} \leqslant 1$ and $dh/dt > 0$ for $A = 1$. If (3.4) does not hold, then $dA/dt = dh/dt = 0$.

Elastoplastic shear strains are described by the equations [4, 6]

$$\frac{ds}{dt}-\tau\left(\frac{\partial u_1}{\partial x_2}-\frac{\partial u_2}{\partial x_1}\right)+\lambda s=\mu\left(\frac{\partial u_1}{\partial x_1}-\frac{\partial u_2}{\partial x_2}\right)$$
$$\frac{d\tau}{dt}+s\left(\frac{\partial u_1}{\partial x_2}-\frac{\partial u_2}{\partial x_1}\right)+\lambda\tau=\mu\left(\frac{\partial u_1}{\partial x_2}+\frac{\partial u_2}{\partial x_1}\right) \tag{3.5}$$

where $s = \frac{1}{2}(\sigma_{11}-\sigma_{22})$, $\tau=\sigma_{12}$, the multiplier $\lambda = 0$ if (3.3) does not hold and is expressed in terms of s and τ when (3.3) is satisfied and $\mu = \mu(A, h)$ is the elastic shear modulus.

Some plastic effects associated with the description of the slip-line configuration in an ice cover under quasisteady loads have been analysed in [4]. In this paper, the main focus is on condensation processes in an ice cover under compression. Hummocking is not considered and we assume that (3.1) does not hold and $dh/dt = 0$. Strain irreversibility during condensation is associated with the breaking-up of small ice floes and of the edges of large ice floes under compression. The broken ice fragments are squeezed out into clear water spaces and onto the surface of surrounding ice floes, slightly changing their thickness. This process can be compared to the destruction of the soil skeleton under compression [6].

The approximate dependence of the pressure p on the strain deviator is shown in Fig. 1 in the form $p = p(\rho, A)$ for $\pi_b = 0$. The feasibility of this representation follows from (2.6) and (2.7). The curve M_*QM_{**} is described by the equation

$$p=\pi_p(A) \tag{3.6}$$

The quantity $\pi_p(A)$ is the pressure at which plastic condensation of a compacted ice cover with compactness A occurs.

Assume that some element of a dispersed ice cover changes to a compacted state when $A = A_*$ at the point M_* (Fig. 1). The compactness increases under compression, and for an arbitrary point Q of the curve M_*QM_{**} we have the condition

$$\rho=\rho_{cr}(A_Q),\quad A_Q>A_*. \tag{3.7}$$

As the load decreases, the ice cover behaves elastically and its compactness A_Q does not change. Load reduction to $p = 0$ corresponds to the curve QQ_0 in Fig. 1:

$$p=\pi(A_Q,\ \rho) \tag{3.8}$$

From (3.6)–(3.8) it follows that ρ_{cr} is given by the equation

$$\pi_p(A_Q)=\pi(A_Q,\ \rho_{cr}) \tag{3.9}$$

The ice cover does not resist further tension and directly passes into the dispersed state.

If tension changes to compression, the ice cover changes from the dispersed to compacted state for $A = A_Q$. Thus, for ice in the state reached by load reduction from the point Q, we have

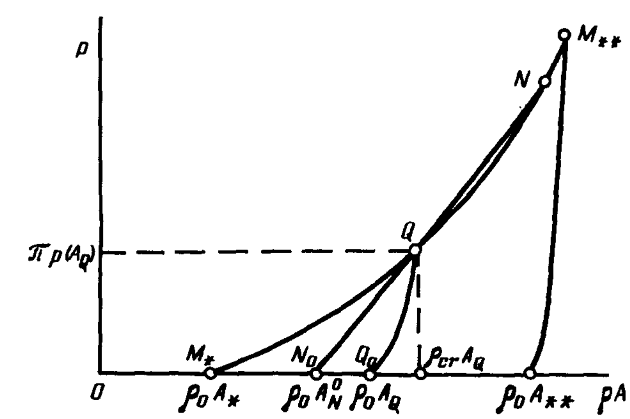

FIG. 1.

$A_* = A_Q$. Further compression obeys the law (3.8) up to the point (3.9), after which the ice cover undergoes plastic condensation according to (3.6). Under pure compression, hummocking starts at the point M_{**} for $p = \pi_p(A_{**})$, $A = A_{**} \leqslant 1$.

We will assume that the functions $\pi_p(A)$, $\pi(A, \rho)$ satisfy the conditions

$$\frac{d^2\pi_p}{dA^2} \geqslant 0, \quad \frac{d\pi_p}{dA} \geqslant 0, \quad \frac{\partial^2\pi}{\partial\rho^2} \geqslant 0, \quad \frac{\partial\pi}{\partial\rho} \geqslant 0 \tag{3.10}$$

The first two conditions in (3.10) follow from the definition of the tensor σ_{ij} given after Eqs (2.3) and from intuitive considerations, which indicate that as the compactness increases the mean contact area between ice floes increases, lower stress concentration is observed under compression and the ice floes can withstand higher loads $\mathbf{F}_n$.

The pressure $\pi_p(A, h)$ at which irreversible strains occur in the ice cover was introduced in [3]. The following empirical formula was proposed for this pressure based on a comparison of numerical calculations with observations in nature [3]:

$$\pi_p(A, h) = p^* h \exp[20(A-1)] \tag{3.11}$$
$$p^* = 5 \times 10^3 \text{ N/m}^2$$

The curve (3.11) obviously satisfies the first two conditions in (3.10). To prove the last two conditions in (3.10), note that in a compacted ice cover ice floes may have contacts in the plane $z = 0$ either along sections of finite length or at isolated points. If we assume that the ice floes show elastic behaviour under rapid loading, then for a point contact between two ice floes we have [7]

$$F = kl^{3/2} \tag{3.12}$$

where F is the compressive force, l is the distance to which the ice floes approach under the action of F and k is a coefficient of proportionality that depends on the geometry of the ice floes near the point of contact and their elastic constants.

From (3.12) it follows that the macroscopic pressure in the ice cover, consisting of elastic ice floes with point contacts, is of the order of $\varepsilon^{3/2}$, where ε is the deviator of the macroscopic strain tensor.

If the ice floes have surface contact, we may assume that the dependence $p(\varepsilon)$ is linear under small strains.

Thus, for small elastic compressive strains of the ice cover in the plane $z = 0$, we can propose the formula

$$\pi(A, \rho) = k_1(A)(\rho - \rho_0)^{3/2} + k_2(A)(\rho - \rho_0) \tag{3.13}$$

The coefficients $k_1(A)$, $k_2(A)$ depend also on the size, the shape, and the elastic constants of the ice floes and on the frequency of point and surface contacts between the ice floes. The function (3.13) obviously satisfies the last two conditions in (3.10).

In the limiting case $A = 1$, we can assume Hooke's law for the generalized plane stress state [8]

$$\pi(1, \rho) = \frac{Eh}{1 - \nu^2} \frac{\rho - \rho_0}{\rho_0} \tag{3.14}$$

where E and ν are Young's modulus and Poisson's ratio of ice. These constants are of the order of [9]

$$E \approx 10^9 \text{ N/m}^2, \qquad \nu \approx 0.3 \tag{3.15}$$

We see from the above scheme that the stress–strain state of compacted ice and the conditions for its transition to a dispersed state under compressive-tensile loads are determined by specifying the functions (3.6) and (3.8) and the initial state in the plane $(p, \rho A)$, i.e. the quantities A_* and A_{**}.

4. We have identified two states of the ice cover: dispersed and compacted. It is interesting to examine the conditions when the transition from one state to another and the change of ice parameters occur in a discontinuous jump within the framework of the rheological model proposed in Sec. 3.

Considering one-dimensional problems of the dynamics of ice of a homogeneous thickness $h = \text{const}$ with time constants $T \ll \Omega^{-1}$ in Eqs (2.3), we may set $d'/dt = d/dt$ and write these equations in the form

$$\frac{\partial}{\partial t}(\rho A) + \frac{\partial}{\partial x}(\rho A u) = 0$$
$$A\rho h\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x}\right) = -\frac{\partial \sigma}{\partial x} + FA \tag{4.1}$$
$$\sigma = p(\rho, A)$$

The system of equations (4.1) with relationships (3.6)–(3.9) is closed.

The relationships across the discontinuity for (4.1) are written in the form [8]

$$\rho_- A_-(u_- - D) = \rho_+ A_+(u_+ - D)$$
$$p_- + h\rho_- A_- u_-(u_- - D) = p_+ + h\rho_+ A_+ u_+(u_+ - D) \tag{4.2}$$
$$p_\pm = p(\rho_\pm, A_\pm)$$

i.e. in the form of laws of conservation of mass and momentum, D is the velocity of propagation of the discontinuity and the subscripts plus and minus are assigned to the variables on the right- and left-hand edges of the discontinuity.

From conditions (4.2) we obtain

$$D = u_- \pm D^* \tag{4.3}$$

$$u_+ = u_- \pm (1 - \rho_- A_-/(\rho_+ A_+))D^*(A_\pm, \rho_\pm) \tag{4.4}$$

$$D^{*2} = \frac{A_+\rho_+}{hA_-\rho_-}\frac{p_+ - p_-}{\rho_+ A_+ - \rho_- A_-}$$

From (3.10) it follows that $D^{*2} \geq 0$.

In the coordinate system attached to the discontinuity, we obtain

$$u_- = \pm D^*, \quad u_+ = \pm \rho_- A_- D^*/(\rho_+ A_+) \tag{4.5}$$

For $\rho_- A_- < \rho_+ A_+$, the upper sign in (4.5) corresponds to condensation discontinuities and the lower sign to rarefaction discontinuities. For $\rho_- A_- > \rho_+ A_+$ the reverse correspondence applies.

Passing to the limits as $\rho_+ A_+ \to \rho_- A_-$ in (4.3) and (4.4), we obtain

$$D = u \pm c, \quad c^2 = h^{-1} dp/d(\rho A) \tag{4.6}$$

(c is the velocity of propagation of small perturbations in an ice cover at rest). From (3.10) and (4.6) it follows that c^2 increases as ρA increases along the plastic and purely elastic strain branches, which are defined by the functions π and π_p. Elastic and plastic condensation discontinuities are therefore evolutionary, whereas the rarefaction discontinuities are unstable [10, 11].

The law of conservation of energy across the discontinuity is written in the form

$$p_- u_- + {}^1/_2 h\rho_- u_-^2 A_-(u_- - D) + e = p_+ u_+ + {}^1/_2 h\rho_+ u_+^2 A_+(u_+ - D) \tag{4.7}$$

where e is the energy released or absorbed in the discontinuity. From (4.5) and (4.7) we obtain

$$e = \pm {}^1/_2 D^*(p_+ + p_-)(\rho_- A_- - \rho_+ A_+)/(\rho_+ A_+) \tag{4.8}$$

We see that condensation discontinuities propagate with release of energy, while rarefaction discontinuities absorb energy for their propagation. The energy released by a condensation discontinuity causes ice breakage, produces chaotic motion of the ice, and is converted into heat. Our analysis indicates that only condensation discontinuities are possible in nature.

Formulas (4.4) for purely elastic discontinuities in a compacted ice are written, apart from terms of the order of $O((\rho_+ - \rho_-)/\rho_0)$, in the form

$$R_1:\ u_+=u_-+\rho_0^{-1}(\rho_+-\rho_-)D^*(A,\ \rho_\pm) \qquad (4.9)$$
$$D^{*2}=h^{-1}(\pi_+-\pi_-)/(\rho_+-\rho_-),\quad \pi_\pm=\pi(A,\ \rho_\pm)$$

For plastic condensation discontinuities, we may assume that strains associated with density changes are small compared with strains associated with compaction changes. From (4.4) we thus obtain

$$R_2:\ u_+=u_-\pm(1-A_-/A_+)D_p{}^*(A_\pm) \qquad (4.10)$$

$$D_p^{*2}=\frac{A_+}{\rho_0 hA_-}\frac{\pi_{p,+}-\pi_{p,-}}{A_+-A_-},\quad \pi_{p,\pm}=\pi_p(A_\pm)$$

From (4.9) and (4.10) we obtain for the velocities of elastic and plastic waves of low intensity, by passing to the limit $\rho_+\to\rho_-$, $A_+\to A_-$

$$c^2=\frac{1}{hA}\frac{\partial\pi(\rho,A)}{\partial\rho},\quad c_p{}^2=\frac{1}{\rho_0 h}\frac{d\pi_p(A)}{dA} \qquad (4.11)$$

Let us estimate the velocities c and c_p from formulas (3.7), (3.10) and (3.11)

$$c(\rho,\ A=1)\approx 10^2\sqrt{10}\ \text{m/s},\quad c_p(A)\approx\sqrt{60}\exp[20(A-1)]\ \text{m/s}$$

We see that in the absence of point contacts between ice floes, $c_p \ll c$.

Let us consider the discontinuity between dispersed and compacted ice. Suppose that the dispersed ice is located to the left of the discontinuity. In formulas (4.4) we thus put

$$p_-=0,\quad \rho_-=\rho_0$$

If the compactness of the dispersed ice increases across this discontinuity to A_*, then making the same assumptions as for type-R_2 discontinuities we obtain from (4.4)

$$R_3:\ u_+=u_-\pm(1-A_-/A_*)D_r{}^*(\rho_+,\ A_*,\ A_-)$$

$$D_r^{*2}=\frac{A_*}{\rho_0 hA_-}\frac{\pi(A_*,\rho_+)}{A_*-A_-}$$

For $\rho_+=\rho_{cr}$ we obtain a type-R_3 discontinuity of maximum intensity:

$$D_4:\ u_+=u_-\pm(1-A_-/A_*)D_{cr}{}^*(A_*,\ A_-)$$

$$D_{cr}^{*2}=\frac{A_*}{\rho_0 hA_-}\frac{\pi_p(A_*)}{A_*-A_-}$$

Strong condensation discontinuities may exist in a dispersed ice cover. Assume that the compactness of the ice cover is $A_-<A_Q$ and $A_*=A_Q$ (Fig. 1). Let us consider the various discontinuities that may link the dispersed state, with parameters ρ_0 and A, with the compacted state with parameters $\rho_{cr}(A_N)$, A_N (the point N in Fig. 1). Draw the line NQ to its intersection with the axis ρA at the point $\rho_0 A_N{}^0$. For $A_-\leqslant A_N{}^0$ the transition from the initial state to the point N may occur by the discontinuity

$$R_5:\ u_+=u_-\pm(1-A_-/A_+)D_p{}^*(A_\pm)$$

$$D_p^{*2}=\frac{A_+}{\rho_0 hA_-}\frac{\pi_p(A_+)}{A_+-A_-}$$

If $A_-\in(A_N{}^0,\ A_Q)$, then the transition from the initial state to the point N cannot be achieved by one discontinuity.

We see from formulas (4.4) that the velocity D of the discontinuity is proportional to the square root of the slope of the segment joining the states before and after the discontinuity on the curve $p=p(\rho,A)$ in Fig. 1. The least velocity of propagation is therefore observed for type-R_3 discontinuities between dispersed and compacted ice. The difference between the ice drift velocities before and after the discontinuity is close to zero and we have the condition $|D^*| \ll |u_\pm|$.

5. Let us consider the problem of the collision of two compacted ice fields at a time $t=0$. We will assume that the contact line of the two fields is along the line $x=z=0$. The parameters of the ice cover are A_1, $u_1>0$, ρ_0, $p_1=0$ for $x<0$, $t=0$ and A_2, $u_2<u_1$, ρ_0, $p_2=0$ for $x>0$, $t=0$. The motion of each ice field is described by the system (4.1). The rheology of the medium is defined by the functions $\pi_1(A, \rho)$, $\pi_{p,1}(A)$, $A_{**,1}$ for $x<0$ and $\pi_2(A,\rho)$, $\pi_{p,2}(A)$, $A_{**,2}$ for $x>0$.

All the conditions of the problem are easily satisfied if in the initial stages of motion we ignore the external forces F in (4.1) and assume that at $t=0$ there is a system of type-R_1 and type-R_2 discontinuities originating from the point $x=0$ whose parameters are related by (4.3), (4.9) and (4.10). The following discontinuity configurations are possible:

1. If the equation

$$\rho_0(u_1-u_2)=(\rho_2^1-\rho_0)D_2^1(\rho_2^1)+(\rho_1^1-\rho_0)D_1^1(\rho_1^1) \tag{5.1}$$
$$(D_1^1(\rho_1^1)=D^*(A_1, \rho_1^1, \rho_0), \quad D_2^1(\rho_2^1)=D^*(A_2, \rho_2^1, \rho_0))$$

has a solution that satisfies the conditions

$$\pi=\pi_1(A_1, \rho_1^1)=\pi_2(A_2, \rho_2^1)\leqslant \min_i(\pi_{p,i}(A_i)), \quad i=1, 2 \tag{5.2}$$

then at $t=0$ two type-R_1 discontinuities originate from the point $x=0$.

This case is shown in Fig. 2. The lines r_1^1, r_2^1 correspond to the two discontinuities and are described by the equations

$$x=D_{1,2}t, \quad D_i=u_i+(-1)^iD_i^1(\rho_i^1) \tag{5.3}$$

The line r is the border between the ice arriving in opposite directions from the two x half-axes and it is described by the equation

$$x=ut \tag{5.4}$$

where u is the drift velocity of the ice inside the angles α_2, β_2. The compactness of the ice is A_1 inside the angles $\alpha_{1,2}$ and A_2 inside the angles $\beta_{1,2}$. The pressure π and the drift velocity u in the region α_2, β_2 are determined from relationships (4.9), (5.1) and (5.2). The trajectories of the ice cover elements are shown by the dashed lines in Fig. 2.

2. Suppose Eq. (5.1) has a solution, but conditions (5.2) are not satisfied and $\min_i(\pi_{p,i}(A_i))=\pi_{p,1}(A_i)$. In this case, the ice cover arriving from $x<0$ is compacted on collision.

If the equation

$$u_1-u_2=((\rho_2^1-\rho_0)D_2^1(\rho_2^1)+(\rho_{cr,1}-\rho_0)\times \tag{5.5}$$
$$\times D_1^1(\rho_{cr,1}))/\rho_0+(A_1^2-A_1)D_1^2(A_1^2)/A_1^2$$
$$(D_1^2(A_1^2)=D_p^*(A_1, A_1^2))$$

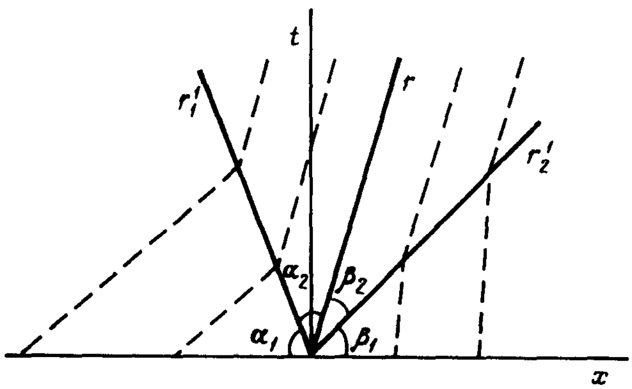

FIG. 2.

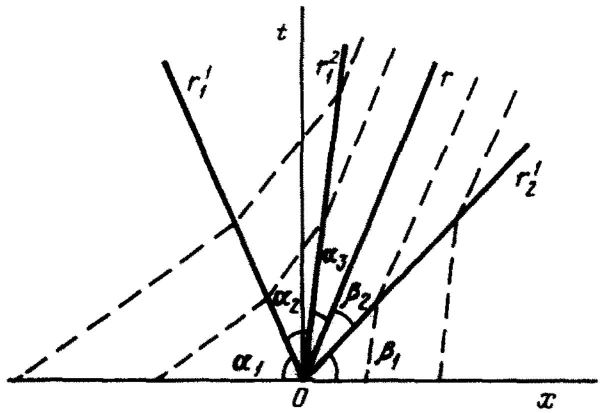

FIG. 3.

has a solution that satisfies the conditions

$$\pi=\pi_{p,1}(A_1^2)=\pi_2(A_2,\ \rho_2^1)\leqslant\pi_{p,2}(A_2) \tag{5.6}$$

that at $t = 0$ two type-R_1 discontinuities and one type-R_2 discontinuity originate from the point $x = 0$.

This case is shown in Fig. 3. The lines r_1^1, r_2^1 correspond to the type-R_1 discontinuities and are described by the equations (5.3), where

$$D_1=u_1-D_1^1(\rho_{cr,1}),\quad D_2=u_2+D_2^1(\rho_2^1)$$

The line r_1^2 corresponds to a type-R_2 discontinuity and is described by the equation $x = (u_1^1 - D_1^2(A_1^2))t$, where u_1^1 is the drift velocity in the region α_2. The line r is described by Eq. (5.4), where u is the drift velocity in the regions α_3, β_2. The compactness of the ice cover is A_1 and A_2 in the regions $\alpha_{1,2}$ and $\beta_{1,2}$, respectively; the compactness in α_3 is A_1^2. The densities in α_2 and β_2 are $\rho_{cr,1}$ and ρ_2^1, respectively. The pressure in α_2 is $\pi_{p,1}(A_1^2)$, the pressure in α_3 is $\pi_{p,1}(A_1^2)$, and the pressure in β_2 is $\pi_2(\rho_2^1)$. The values of u, A_1^2, ρ_2^1 are determined from (4.9), (4.10), (5.5) and (5.6).

3. If conditions (5.2) are not satisfied and $\min_i(\pi_{p,i}(A_i)) = \pi_{p,2}(A_2)$, then the ice cover arriving from the positive x half-axis is compacted on collision. This case is analysed in the same way as case 2.

4. Assume that Eq. (5.5) has a solution, but conditions (5.2) and (5.6) are not satisfied. In this case, the ice cover arriving from the directions of both x half-axes is compacted on collision.

If the equation

$$\begin{gathered}u_1-u_2=((\rho_{cr,2}-\rho_0)D_2^1(\rho_{cr,2})+(\rho_{cr,1}-\rho_0)\times\\ \times D_1^1(\rho_{cr,1}))/\rho_0+(A_2^2-A_2)D_2^2(A_2^2)/A_2^2+\\ +(A_1^2-A_1)D_1^2(A_1^2)/A_1^2\\ (D_2^2(A_2^2)=D_p{*}(A_2,\ A_2^2))\end{gathered} \tag{5.7}$$

has a solution that satisfies the conditions

$$\pi=\pi_{p,1}(A_1^2)=\pi_{p,2}(A_2^2)\leqslant\min_i(\pi_{p,i}(A_{**,i})) \tag{5.8}$$

then at $t = 0$ two type-R_1 discontinuities and two type-R_2 discontinuities originate from the point $x = 0$.

This case is shown in Fig. 4. The lines r_1^1, r_2^1 correspond to the type-R_1 discontinuities and are described by Eqs (5.3), where

$$D_i=u_i+(-1)^iD_i^1(\rho_{cr,i})$$

The lines r_1^2, r_2^2 correspond to the type-R_2 discontinuities and are described by the equations

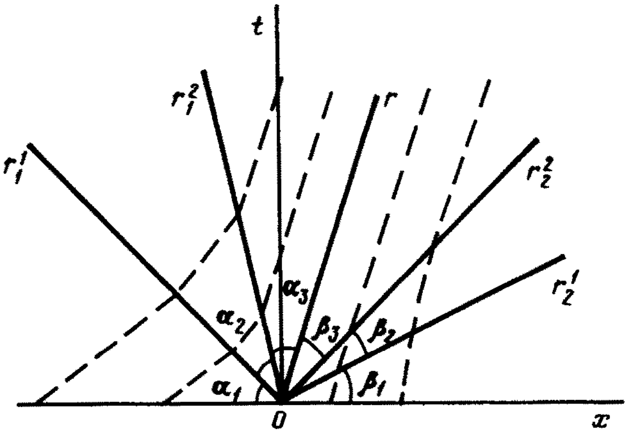

FIG. 4.

$$x=(u_i^1+(-1)^i D_i^2(A_i^2))t$$

where u_1^1 and u_2^1 are the drift velocities inside the angles α_2 and β_2, respectively. The line r is described by Eq. (5.4), where u is the drift velocities inside the angles α_3 and β_3. The compaction, the density and the pressure in $\alpha_{1,2}$ and $\beta_{1,2}$ are A_1, $\rho_{\mathrm{cr},1}$, $\pi_{p,1}(A_1)$ and A_2, $\rho_{\mathrm{cr},2}$, $\pi_{p,2}(A_2)$, respectively. The compactness and the pressure in α_3 and β_3 are A_1^2, $\pi_{p,1}(A_1^2)$ and A_2^2, $\pi_{p,2}(A_2^2)$, respectively. The values of u_1^1, u_2^1, u, A_1^2, A_2^2 are determined from the relationships (4.9), (4.10), (5.7) and (5.8).

If condition (5.8) is not satisfied, the colliding ice floes form into hummocks.

The problems of the collision of two dispersed ice fields and of a dispersed ice field with a compacted field are analysed similarly. The discontinuity configurations obtained in these cases include waves of types R_1–R_5.

Note that when two dispersed ice fields of low compactness collide, a complex drift pattern may develop in the interaction region, with different ice floes acquiring widely differing velocities. Our model is therefore inapplicable in this case. For sufficiently high compactness of the colliding fields, a compacted ice region may form near the contact line.

6. Let us consider in more detail the collision of two dispersed ice fields, for which the functions $\pi_p(A)$, $\pi(A, \rho)$ and the quantities A_* and A_{**} are equal. The parameters of the ice arriving from the negative x half-axis are given a subscript of one and the parameters of the ice arriving from the positive x half-axis are given a subscript of two. We assume that $u_1>0$, $u_1>u_2$ and the equation

$$(u_1-u_2)A_*=(A_*-A_1)\overset{*}{D}_{r,1}+(A_*-A_2)\overset{*}{D}_{r,2},\qquad \overset{*}{D}_{r,i}=D_r{}^*(\rho, A_*, A_i) \tag{6.1}$$

has a solution that satisfies the condition

$$\pi(\rho, A_*)\leqslant\pi_p(A_*) \tag{6.2}$$

In this case, a narrow strip of ice with compactness A_* is formed after collision; the boundaries of this strip are lines of discontinuity of type R_3. An illustration of this case is given in Fig. 2, with the lines r_1^1, r_2^1, r described by Eqs (5.3) and (5.4), where $D_i=u_i+(-1)^i D^*_{r,i}$ and u is the drift velocity inside the angles α_2 and β_2. The compactness, the density, and the pressure inside α_2 and β_2 are A_*, ρ, $\pi(\rho, A_*)$, respectively. The drift velocity is determined from the relationships on the R_3 discontinuity:

$$u=u_1-\overset{*}{D}_{r,1}(1-A_1/A_*) \tag{6.3}$$

With time, the pattern of motion will change, as the effect of the external forces on the ice cover are felt. We write these forces in the form

$$F=\rho_a C_a(A)V^2-\rho_w C_w u \tag{6.4}$$

The first term is the eddy friction of the wind with the ice-covered water surface [12], where ρ_a, C_a, V are the

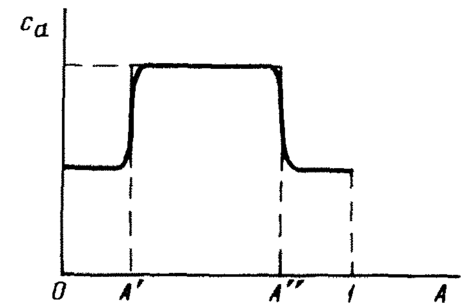

FIG. 5.

air density, the coefficient of friction and the wind velocity; the second term is the Newtonian friction exerted on the ice cover by the bulk of the water at rest [12], where ρ_w, C_w are the water density and the coefficient of friction.

The form of the function $C_a(A)$ based on experimental data [13] is shown in Fig. 5. The shape of the curve $C_a(A)$ is explained by the fact that the roughness of the water surface totally free from ice or totally covered with ice is less than the roughness of the water surface partially covered with ice.

From relationships (4.1), (6.4), and (6.5) it follows that the steady drift velocity of an ice cover with homogeneous compactness A = const is given by

$$u_s(A, V) = \frac{\rho_a C_a(A)}{\rho_w C_w} V^2, \quad V = \text{const}$$

Hence we see that the collision of two ice fields may be caused by the different effect of wind velocities on ice with different compactness values. We accordingly assume in our problem

$$u_i = u_s(A_i, V), \quad A_1 \in (A', A''), \quad A_2 \in (A'', 1)$$

Collision produces a strip of compacted ice with compactness $A_* \in (A'', 1)$. The initial velocity u of this strip is defined by relationships (6.3) and falls in the range (u_1, u_2). The water thus exerts a Newtonian friction on the ice strip, which acts to reduce its velocity.

We will write the law of conservation of momentum for the strip as a whole and the laws of conservation of mass on the two sides of the strip $r_1{}^1$ and $r_2{}^1$:

$$\begin{gathered} d(uA_*(L_2-L_1))/dt = A_1u_1(u_1-D) + \\ + A_2u_2(D_2-u_2) + (\rho_0 h)^{-1}(L_2-L_1)(\rho_a C_a(A_*)V^2 - \rho_w C_w u) \\ A_i(D_i-u_i) = A_*(D_i-u) \\ dL_i/dt = D_i, \quad i=1,2 \end{gathered} \tag{6.5}$$

Hence we obtain

$$\frac{\rho_0 h A_*}{\rho_w C_w}\frac{du}{dt} = u_s(A_*, V) - u \tag{6.6}$$

As the initial value $u(t=0)$ we take (6.3).

From Eq. (6.6) it follows that the velocity of the strip eventually tends to $u_s(A_*, V) = u_2$. This means that the intensity of the discontinuity $r_2{}^1$ tends to zero and the intensity of the discontinuity $r_1{}^1$ tends to a constant value, which is defined by the relationship for R_3 with $u_- = u_1$, $A_- = A_1$.

7. Consider the problem of condensation of the ice cover near a solid wall. Suppose that an ice cover with constant drift velocity $u>0$ and compactness A meets a solid obstacle at the point $x=0$ at time $t=0$, and its velocity instantaneously drops to zero at the point $x=0$. By the law of conservation of momentum, the loss of momentum leads to an abrupt increase in the pressure exerted by the ice cover on the wall, which in every special case is defined by Eqs (7.1), (7.3), (7.5), (7.7), and (7.9) given below.

Assume that the ice cover is in the compacted state, i.e. $A = A_*$. If the equation

$$u = \frac{1}{\rho_0}\left(\frac{(\rho_1-\rho_0)\pi(\rho_1, A)}{h}\right)^{1/2} \tag{7.1}$$

has a solution that satisfies the condition

$$\pi(\rho_1, A) \leqslant \pi_p(A) \tag{7.2}$$

then a type-R_1 discontinuity originates from the point $x = 0$ at $t = 0$.

If the equation

$$u = \frac{1}{\rho_0}\left(\frac{(\rho_{cr}-\rho_0)\pi_p(A)}{h}\right)^{1/2} + \left(\frac{(A_1-A)(\pi_p(A_1)-\pi_p(A))}{\rho_0 hAA_1}\right)^{1/2} \tag{7.3}$$

has a solution that satisfies the condition

$$A_1 \leqslant A_{**} \tag{7.4}$$

then a type-R_1 discontinuity and a type-R_2 discontinuity originate from the point $x = 0$ at $t = 0$. In this case, the ice cover is condensed near the wall.

If condition (7.4) is not satisfied, hummocking occurs near the wall. By measuring the pressures π and π_p and the velocity u, we can use relationships (7.1) and (7.3) to determine the dependences $\rho_{cr}(A)$, $\pi_p(A)$, $\pi(\rho, A)$.

Assume that the ice cover drifting to the wall is in a dispersed state with compactness $A < A_*$. If the equation

$$u = \left(\frac{(A_*-A)\pi(\rho, A_*)}{\rho_0 hAA_*}\right)^{1/2} \tag{7.5}$$

has a solution that satisfies the condition

$$\pi(\rho, A_*) < \pi_p(A_*) \tag{7.6}$$

then a type-R_3 discontinuity originates from the point $x = 0$ at $t = 0$.

If condition (7.6) does not hold, then condensation of the ice cover occurs near the wall.

If the equation

$$u = \left(\frac{(A_1-A)\pi_p(A_1)}{\rho_0 hAA_1}\right)^{1/2} \tag{7.7}$$

has a solution that satisfies the conditions

$$A_1 \leqslant A_{**}, \quad \frac{\pi_p(A_1)}{A_1-A} \leqslant \frac{\pi_p(A_1)-\pi_p(A_*)}{A_1-A} \tag{7.8}$$

then a type-R_5 discontinuity originates from the point $x = 0$ at $t = 0$.

If the second condition in (7.8) does not hold and the first condition is satisfied, then the pressure on the wall $\pi_p(A_1)$ is obtained by solving the equation

$$u = \left(\frac{(A_*-A)\pi_p(A_*)}{\rho_0 hAA_*}\right)^{1/2} + \left(\frac{(A_1-A_*)(\pi_p(A_1)-\pi_p(A_*))}{\rho_0 hA_1A_*}\right)^{1/2} \tag{7.9}$$

In this case, a type-R_4 discontinuity originates from the point $x = 0$ at $t = 0$, followed by a type-R_2 discontinuity. The compactness of the ice cover rises to A_* behind the first discontinuity and to A_1 behind the second discontinuity.

If the first condition in (7.8) does not hold, hummocking of the ice cover occurs near the obstacle.

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Translated by Z.L.

J. Appl. Maths Mechs Vol. 56, No. 3, pp. 358–364, 1992
Printed in Great Britain.
0021–8928/92 $15.00 + .00

DUAL ESTIMATES IN PROBLEMS OF DESIGN OF ELASTIC STRUCTURES†

L. V. PETUKHOV

St Petersburg

(*Received* 28 *January* 1991)

The problem of minimizing the volume of two- and three-dimensional structures, subject to certain stress constraints, which are known as the conditions of strength theory and are used in practice for various materials, is considered. The control is achieved by adjusting the shape of the boundary. Cavities inside the design region are allowed, and the shape of the cavities is also optimized. Dual problems, constructed for such optimal design problems, can be used for estimates of optimal or nearly optimal designs. Examples of dual estimates for three optimal design problems are considered.

1. STATEMENT OF THE PROBLEM

WE HAVE previously introduced [1] the notions of the design region and the feasible region, and proved existence theorems for the first and second variations of the displacements of the elastic region. We denote by $O(s, \lambda)$ the set of feasible regions $\Omega \subset \Omega^\circ$, where Ω° is the design region (here $0 < \lambda < 1$ and s is an integer characterizing the smoothness of the boundary Γ of the region Ω [1]).

Let us formulate the optimal design problem. Suppose we are given the shear modulus μ, Poisson's ratio ν, and the yield point σ_0 of the material, the external load vector $\mathbf{F}$ acting on the part of the boundary Γ_F°, and the section of the boundary Γ_u°, where the displacements of the elastic region are zero. It is required to find

$$\inf J(\mathbf{u}), \quad J = \int_\Omega dx, \quad \forall \Omega \in O(s, \lambda) \tag{1.1}$$

where $\mathbf{u} = u_i \mathbf{e}_i$ is the solution of the integral identity

† *Prikl. Mat. Mekh.* Vol. 56, No. 3, pp. 434–441, 1992.